The third stage of learning math facts:

Developing automaticity

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ABSTRACT

Learning math facts proceeds through three stages: 1) procedural knowledge of figuring out facts; 2) strategies for remembering facts based on relationships; 3) automaticity in math facts—declarative knowledge. Students achieve automaticity with math facts when they can directly retrieve the correct answer, without any intervening thought process. The development of automaticity is critical so students can concentrate on higher order thinking in math. Students who are automatic with math facts answer in less than one second, or write between 40 to 60 answers per minute, if they can write that quickly. Research shows that effective math facts practice proceeds with small sets of no more than 2 –4 facts at a time. During practice, the answers must be remembered rather than derived. Practice must limit response times and give correct answers immediately if slow. Automaticity must be developed with each set of facts, and maintained with the facts previously mastered, before more facts are introduced. Suggestions for doing this with flashcards or with worksheets are offered.

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"Everyone knows what automaticity is. It is the immediate, obligatory way we apprehend the world around us. It is the fluent, effortless manner in which we perform skilled behaviors. It is the 'popping into mind' of familiar knowledge at the moment we need it (Logan, 1991a, p. 347)."

Researchers have long maintained there are three general stages of "learning" math facts and different types of instructional activities to effect learning in those stages (Ando & Ikeda, 1971; Ashlock, 1971; Bezuk & Cegelka, 1995; Carnine & Stein, 1981; Garnett, 1992; Garnett & Fleischner, 1983). Researchers such as Garnett (1992) have carefully documented the development sequence of procedures that children use to obtain the answers to math facts. Children begin with a first stage of counting, or procedural knowledge of math facts. A second stage consists of developing ways to "remember" math facts by relating them to known facts. Then the third and final stage is the declarative knowledge of the "knowing" the facts or "direct retrieval." Another line of research has established that most children move from using procedures to "figure out" math facts in the first grade to the adult model of retrieval by fifth grade (Koshmider & Ashcraft, 1991). The switch from predominantly procedural to predominantly declarative knowledge or direct retrieval of math facts normally begins at around the 2nd to 3rd grade level (Ashcraft, 1984).

If these three levels of knowing math facts are attainable, what does that imply about how to teach math facts? Different teaching and practice procedures apply to each of these stages. Meyers & Thorton (1977) suggested activities specifically geared toward different levels, including activities for “overlearning” intended to promote rapid recall of a set or cluster of facts. As Ashcraft (1982) points out there is much more evidence of direct retrieval of facts when you study adults or children in 4th grade or above. Examining only primary age children has led to summaries of fact solving strategies that may give too much weight to soon-to-be-discarded methods of answering math facts. Much of the research on the efficacy of deriving facts from relationships was developed in studies on addition and subtraction with primary age children (Carpenter & Moser, 1984; Garnett, 1992; Steinberg, 1985; Thorton, 1978).

Much of the advice on “how to teach” math facts is unclear about which stage is being addressed. The term “learning” has been variously applied to all three stages, and the term “memorization” has been used to apply to both the second and third stages (Stein, Silbert & Carnine, 1997). What is likely is that teaching strategies that address earlier stages of learning math facts may be counterproductive when attempting to develop automaticity. This paper will examine research on math fact learning as it applies to each of the three stages, with an emphasis on the third stage, the development of automaticity.

First stage: Figuring out math facts

The first stage has been characterized as “understanding,” or “conceptual” or “procedural.” This is the stage where the child must count or do successive addition, or some other strategy to “figure out” the answer to a fact (Garnett, 1992). “Specifically in view is the child’s ability to associate the written number sentence with a physical referent (Ashlock, 1971, p. 359). There is evidence that problems with learning mathematics may have their origins in failure to completely master counting skills in kindergarten and first grade (Geary, Bow-Thomas, & Yao, 1992). Students do need to have mastered a procedure for computing simple facts. At the beginning stage arithmetic facts are problems to be solved (Gersten & Chard, 1999). If students cannot solve basic fact problems given plenty of time—then they simply do not understand the process, and certainly are not ready to begin memorization (Kozinski & Gast, 1993). Students must understand the process well enough to solve any fact problem given them before beginning memorization procedures. "To ensure continued success and progress in mathematics, students should be taught conceptual understanding prior to memorization of the facts (Miller, Mercer, & Dillon, 1992, p. 108)." Importantly, the conceptual understanding, or procedural knowledge of counting, is, prior to, rather than, in place of, memorization. "Prior to teaching for automaticity, however, it is best to develop the conceptual understanding of these math facts as procedural knowledge (Bezuk & Cegelka, 1995, p. 365)."

Second stage: Strategies for remembering math facts

The second stage has been characterized as "relating" or as "strategies for remembering." This can include pairs of facts related by the commutative property, e.g., 5 + 3 = 3 + 5 = 8. This can also include families of facts such as 7 +4 = 11, 4 + 7 = 11, 11 - 4 = 7, and 11 – 7 = 4. Garnett characterizes such strategies as more "mature" than counting procedures indicating, "Another mature strategy is 'linking' one problem to a related problem (e.g., for 5 + 6, *thinking '5 + 5 =10, so 5 + 6 = 11'*) (Garnett, 1992, p.

212). The goal of the second stage is the development of accuracy rather fluency or automaticity. Studies looking at use of “strategies for remembering” seldom use timed tests, and never have rigorous expectations of highly fluent performance. As long as students are accurate the strategy is considered successful.

Carpenter and Moser (1984) reported on a longitudinal study of children’s evolving methods of solving simple addition and subtraction problems in first through 3rd grade. They identified sub-stages within counting as well as variations on recall strategies. The focus of their study was on naturalistically developed strategies that children had made up to solve problems they were presented, rather than evaluating a teaching sequence for its efficiency. For example, they noted that children were not consistent in using the most efficient strategy, even when they sometimes used that method. The curriculum used in all the classrooms studied was “Developing Mathematical Processes” which focused on problem solving using manipulatives, rather than direct teaching of algorithms. However Carpenter and Moser did caution that, “It has not been clearly established that instruction should reflect the natural development of concepts and skills (1984, p. 200).” Garnett (1992) identified similar stages and sub-stages in examining children with learning disabilities.

There has been much research on teaching children some of the relationships among facts or groups of facts. Many studies have been reported in which various rules and relationships are taught to help children “derive” the answers to math facts (Carnine & Stein, 1981; Rightsel & Thorton, 1985; Steinberg, 1985; Thorton, 1978; Thorton & Smith, 1988; Van Houten, 1993).

An example of such a relationship rule is the, "doubles plus 1" rule. This rule applies to facts such as 8 + 9 which is the same as 8 + 8 plus 1, therefore because 8 + 8 = 16 and 16 + 1 = 17, therefore 8 + 9 = 17 (Thorton, 1978). This is clearly different from "direct retrieval" although some researchers suggest that learning such a strategy is part of memorization. Most authors suggest that learning the facts in terms of relationships is an intermediate stage between conceptual development that involves using a procedure and drill for mastery or automatic recall (Steinberg, 1985; Stein et al., 1997; Suydam, 1984; Thorton, 1978;). "Research has indicated that helping them [students] to develop thinking strategies is an important step *between* the development of concepts with manipulative materials and pictures and the mastery of the facts with drill-and-practice activities [emphasis added] (Suydam, 1984, p. 15). However, these authors state or imply that addition practice activities are needed (after teaching thinking strategies) in order to develop automatic, direct retrieval of facts.

A lot of research on math facts acquisition as well as the theorizing about "how math facts should be taught" has focused on this stage of learning. Researchers have repeatedly found that teaching a set of related facts whose answers can be derived from a rule of some kind are "learned" more quickly than randomly chosen and presented facts (Carnine & Stein, 1981; Van Houten, 1993). However, the term "learned" in these studies indicates accurate responding rather than the development of automaticity—as measured by rapid rates of responding in the realm of 40 problems per minute.

Thorton suggested that "curriculum and classroom efforts should focus more carefully on the development of strategy prior to drill on basic facts (1978, p. 226)." Thorton's (1978) research showed that using relationships such as doubles and doubles plus 1, and other "tricks" to remember facts as part of the teaching process resulted in more facts being learned after eight weeks than in classes where such aids to memory were not taught. Similarly, Carnine and Stein (1981) found that students instructed with a strategy for remembering facts learned a set of 24 facts with higher accuracy than students who were presented the facts to memorize with no aids to remembering (84% vs. 59%).

Steinberg (1985) studied the effect of teaching "noncounting, derived facts strategies in which the child uses a small set of known number facts to find or derive the solution to unknown number facts (1985, p. 337)." The subjects were second graders and the focus was on addition and subtraction facts. The children changed strategies from counting to using the derived facts strategies and did write the answers to more of the fact problems within the 2 seconds allotted per problem than before such instruction. The study did not demonstrate that teaching derived facts strategies led to the recall of number facts.

Thorton and Smith (1988) found that by teaching strategies and relationship activities to first graders found that they correctly answered more facts than a control group. In the timed tests rates were about 20 problems per minute in the experimental group and about 10 per minute in the traditional group. Although rates of responding did not indicate automaticity, approximately 55% of the experimental group reported that rather than using counting strategies, they had mostly "memorized" a set of 9 target subtraction facts compared to 12% in the control group.

Van Houten (1993) specifically taught rules to children (although the rules were unlike those normally used by children, or discussed in the literature) and found that the children "learned" a set of seven facts more rapidly when there was a "rule" to remember them with compared to a random mix of seven facts with no relationships. However, as in the other studies, Van Houten did not measure development of automaticity through some kind of timing criteria. Instead accuracy was the only measure of learning.

Isaacs & Carroll (1999) listed a potential instructional sequence of relationships for addition and subtraction facts:

"1. Basic concepts of addition; direct modeling and 'counting all' for addition
2. The 0 and 1 addition facts; 'counting on'; adding 2
3. Doubles (6 + 6, 8 + 8, etc.)
4. Complements of 10 (9 + 1, 8 + 2, etc.)
5. Basic concepts of subtraction; direct modeling for subtraction
6. Easy subtraction facts (-0, -1, and -2 facts); 'counting back' to subtract
7. Harder addition facts; derived-fact strategies for addition (near doubles, over-10 facts)
8. 'Counting up' to subtract
9. Harder subtraction facts; derived-fact strategies for subtraction (using addition facts, over-10 facts) (Isaacs & Carroll, 1991, p. 511-512)."

With the variety of rules that have been researched and found to be effective, the exact nature of the rule children use is apparently immaterial. It appears that any rule or strategy that allows the child to remember the correct answer, insures that practice will be "perfect practice" and learning will be optimized. Successful students often hit early on a strategy for remembering simple facts, where less successful students lack such a strategy and may simply guess (Thorton, 1978; Carnine & Stein, 1981). Teaching students the facts in some logical order that emphasizes the relationships makes them easier to remember. Performance may deteriorate if practice is not highly accurate; because if the student is allowed to give incorrect answers, those will then “compete” with the correct answer in memory (Goldman & Pellegrino, 1986).

Third stage: Developing automaticity with math facts

The third stage of learning math facts has been called mastery, or overlearning, or the development of automaticity. In this stage children develop the capacity to simply recall the answers to facts without resorting to anything other direct retrieval of the answer. Ashlock (1971) indicated that children must have “immediate recall” of the basic facts so they can use them “with facility” in computation. “If not, he may find it difficult to develop skill in computation for he will frequently be diverted into tangential procedures (1971, p. 363).”

When automaticity is developed, one of its most notable traits is speed of processing. “Proficient levels of performance go beyond the accuracy (quality) of an acquired skill to encompass sufficient speed (quantity) of performance. It is this sort of proficiency with basic facts, rather than accuracy per se, which is so notably lacking in many learning disabled children’s computation performance (Garnett & Fleischner, 1983, p. 224).”

The development of automaticity has been studied extensively in the psychology literature. Research by psychologists on math facts and other variations on automatic fact retrieval has been extensive, probably because “Children’s acquisition of skill at mental addition is a paradigm case of automaticity (Logan & Klapp, 1991a, p.180).

While research continues to refine the details of the models of the underlying processes, there has for some time been a clear consensus that adults “retrieve” the answers to math facts directly from memory. The “modern theories that argue that the process underlying automaticity is memory retrieval: According to these theories, performance is automatic when it is based on direct-access, single-step retrieval of solutions from memory rather than some algorithmic computation (Logan & Klapp, 1991a, p. 179).” When facts have been well practiced, they are “remembered” quickly and automatically—which frees up other mental processes to use the facts in more complex problems (Ashcraft, 1992; Campbell, 1987b; Logan, 1991a).

Baroody made one of the last forceful defenses of the alternative model that adults continue to use, albeit very quickly, “rules, procedures, or principles from which a whole range of combinations could be reconstructed (1985, p. 95).” Baroody’s examples of how we remember the + 0 facts or the + 1 facts are now understood to be exceptions to the general rule of simple retrieval from memory. Baroody then took the exceptions to make a rule and went on to say that, “According to this alternative model, ‘mastery of the facts’ would *include* discovering, labeling, and internalizing *relationships*. Meaningful instruction (the teaching of thinking strategies) would probably contribute more directly to this process than drill approach alone (1985, p. 95).” This did not include any empirical examinations of his theory.

Baroody’s notion suggests that additional work to memorize math facts is not necessary as long as students can reconstruct facts from their relationships with other known facts. However, a variety of careful psychological experiments have demonstrated that adults do not use procedures or algorithms to derive the answers to math facts—they simply remember or retrieve them (Ashcraft, 1985; Campbell, 1987a; Campbell, 1987b; Campbell & Graham, 1985; Graham, 1987; Logan, 1988; Logan & Klapp, 1991a; Logan & Klapp, 1991b; McCloskey, Harley, & Sokol, 1991).

For example, Logan and Klapp (1991a) did experiments with “alphabet arithmetic” where each letter corresponded to a number (A=1, B=2, C=3, etc.). In these studies college students initially experienced the same need to count to figure out sums that elementary children did. In the initial-counting stage the time to answer took longer the higher the addends (requiring more time to count up). However, once these “alphabet facts” became automatic—they took the same amount of time regardless of the addends, and they took less time than they could possibly have counted. In addition, they found that when students were presented with new items they had not previously had—they reverted to counting and the time to answer went up dramatically and was again related to the size of the addends. When the students were presented with a mix of items, some which required counting and others, which they had already learned, they were still able to answer the ones they had previously learned roughly as fast as they had before. Response times were able to clearly distinguish between the processes used on facts the students were “figuring out” vs. the immediate retrieval process for facts that had become automatic.

Does the same change in processes apply to children as they learn facts? Another large set of experiments has demonstrated that as children mature they transition from procedural (counting) methods of solving problems to the pattern shown by adults of direct retrieval. (Ashcraft, 1982; Ashcraft, 1984; Ashcraft, 1992; Campbell, 1987; Campbell & Graham, 1985; Geary & Brown, 1991; Graham, 1987; Koshmider &

Ashcraft, 1991; Logan, 1991b; Pelegrino & Goldman, 1987). Despite the preponderance of clear evidence that adults and children (after they have learned math facts to automaticity) use direct retrieval—"just remembering"—some writers in respected education journals continue to advance Baroody's theory that "although many facts become automatic, adults also use strategies and rules for certain facts (Isaacs & Carroll, 1999, p. 514)." This phenomena is limited to facts adding + 0 or multiplying x 0 or x 1 (Ashcraft, 1985) and therefore does not indicate adult use of such strategies and rules for any of the rest of the facts. Unfortunately it appears as if some of those who recommend teaching strategies and relationships in the second stage of math facts see it primarily as a way to de-emphasize the memorization needed for automatization of math facts (Isaacs & Carroll, 1999).

Interestingly, most of the research celebrating "strategies and rules" relates to addition and subtraction facts. Once children are expected to become fluent in recalling multiplication facts such strategies are inadequate to the task, and memorization seems to be the only way to accomplish the goal of mastery. Campbell and Graham's (1985) study of error patterns in multiplication facts by children demonstrated that as children matured errors became limited to products related correctly to one or the other of the factors, such as 4 X 7 = 24. See also (Campbell, 1987b; Graham, 1987). They concluded that "the acquisition of simple multiplication skill is well described as a process of associative bonding between problems and candidate answers....determined by the relative strengths of correct and competing associations ... not a consequence ... of the execution of reconstructive arithmetic procedures (Campbell & Graham, 1985, p. 359)." In other words, we remember answers based on practice with the facts rather than using some procedure to derive the answers. And we have difficulty learning them because we recall answers to other similar math fact problems and sometimes fail to correctly discard these incorrect answers (Campbell, 1987b).

Graham (1987) reported on a study where students who learned multiplication facts in a mixed order had little or no difference between response times to smaller and larger fact problems. Graham's recommendation was that facts be arranged in sets where commonly confused problems are not taught together. In addition he suggested, "it may be beneficial to give the more difficult problems (e.g. 3 x 8, 4 x 7, 6 x 9, and 6 x 7) a head start. This could be done by placing them in the first set and requiring a strict performance criterion before moving on to new sets (Graham, 1987, p. 139)."

Ashcraft has studied the development of mental arithmetic extensively (Ashcraft, 1982; Ashcraft, 1984; Ashcraft, 1992; Koshmider & Ashcraft, 1991). These studies have examined response times and error rates over a range of ages. The picture is clear. In young children who are still counting, response rates are consistent with counting rates. Ashcraft found that young children took longer to answer facts—long enough to count the answers. He also found that they took longer in proportion to the sum of the addends—an indication that they were counting. After children learn the facts, the picture changes. Once direct retrieval is the dominant mode of answering, response rates decrease from over 3 seconds down to less than one second on average. Ashcraft notes that other researchers have demonstrated that while young children use a variety of approaches to come up with the answer (as they develop knowledge of the math facts), eventually these are all replaced by direct retrieval in 5th and 6th grade normally achieving children.

Ashcraft and Christy (1995) summarized the model of math fact performance from the perspective of cognitive psychology. "...the whole-number arithmetic facts are eventually stored in long-term memory in a network-like structure .... at a particular level of strength, with strength values varying as a function of an individual's practice and experience....because these variables determine strength in memory (Ashcraft & Christy, 1995, p. 398)." Interestingly enough, Ashcraft and Christy's (1995) examination of math texts led to the discovery in both addition and multiplication that smaller facts (2s-5s) occur about twice as frequently as larger facts (6s-9s). This may in part explain why larger facts are less well learned and have slower response times.

Logan's (1991b) studies of the transition to direct retrieval show that learners start the process of figuring out the answer (procedural knowledge) and use retrieval if the answer is remembered (declarative knowledge) before the process is completed. For a time there is a mixture of some answers being retrieved from memory while others are being figured out. Once items are learned to the automatic level, the answer "occurs" to the learner before they have time to count. Geary and Brown (1991) found the same sort of evolving mix of strategies among gifted, normal, and math-disabled children in the third and fourth grades.

Why is automaticity with math facts important?

Psychologists have long argued that higher-level aspects of skills require that lower level skills be developed to automaticity. Turn-of-the-century psychologists eloquently captured this relationship in the phrase, "Automaticity is not genius, but it is the hands and feet of genius." (Bryan & Harter, 1899; as cited in Bloom, 1986).

Automaticity occurs when tasks are learned so well that performance is fast, effortless, and not easily susceptible to distraction (Logan, 1985).

An essential component of automaticity with math facts is that the answer must come by means of direct retrieval, rather than following a procedure. This is tantamount to the common observation that students who "count on their fingers" have not mastered the facts. Why is "counting on your fingers" inadequate? Why isn't procedural knowledge or counting a satisfactory end point? "Although correct answers can be obtained using procedural knowledge, these procedures are effortful and slow, and they appear to interfere with learning and understanding higher-order concepts (Hasselbring, Goin and Bransford, 1988, p. 2)." The notion is that the mental effort involved in figuring out facts tends to disrupt thinking about the problems in which the facts are being used.

Some of the argument of this information-processing dilemma was developed by analogy to reading, where difficulty with the process of simply decoding the words has the effect of disrupting comprehension of the message. Gersten and Chard illuminated the analogy between reading and math rather explicitly. "Researchers explored the devastating effects of the lack of automaticity in several ways. Essentially they argued that the human mind has a limited capacity to process information, and if too much energy goes into figuring out what 9 plus 8 equals, little is left over to understand the concepts underlying multi-digit subtraction, long division, or complex multiplication (1999, p. 21)."

It requires a good deal of practice to develop automaticity with math facts. "The importance of drill on components [such as math facts] is that the drilled material may become sufficiently over-learned to free up cognitive resources and attention. These cognitive resources may then be allocated to other aspects of performance, such as more complex operations like carrying and borrowing, and to self-monitoring and control (Goldman & Pellegrino, 1986, p. 134)." This suggests that even mental strategies or mnemonic tricks for remembering facts must be replaced with simple, immediate, direct retrieval, else the strategy for remembering itself will interfere with the attention needed on the more complex operations. For automaticity with math facts to have any value, the answers must be recalled with no effort or much conscious attention, because we want the conscious attention of the student directed elsewhere.

"For students to be able to recall facts quickly in more complex computational problems, research tells us the students must know their math facts at an acceptable level of 'automaticity.' Therefore, teachers...must be prepared to supplement by providing more practice, as well as by establishing rate criteria that students must achieve. (Stein et al., 1997, p. 93)." The next question is what rate criteria are appropriate?

How fast is fast enough to be automatic?

Some educational researchers consider facts to be automatic when a response comes in two or three seconds (Isaacs & Carroll, 1999; Rightsel & Thorton, 1985; Thorton & Smith, 1988). However, performance is not automatic, direct retrieval when it occurs at rates that purposely "allow enough time for students to use efficient strategies or rules for some facts (Isaacs & Carroll, 1999, p. 513)."

Most of the psychological studies have looked at automatic response time as measured in milliseconds and found that automatic (direct retrieval) response times are usually in the ranges of 400 to 900 milliseconds (less than one second) from presentation of a visual stimulus to a keyboard or oral response (Ashcraft, 1982; Ashcraft, Fierman & Bartolotta, 1984; Campbell, 1987a; Campbell, 1987b; Geary & Brown, 1991; Logan, 1988). Similarly, Hasselbring and colleagues felt students had automatized math facts when response times were "down to around 1 second" from presentation of a stimulus until a response was made (Hasselbring et al. 1987)." If however, students are shown the fact and asked to read it aloud then a second has already passed in which case no delay should be expected after reading the fact. "We consider mastery of a basic fact as the ability of students to respond immediately to the fact question. (Stein et al., 1997, p. 87)."

In most school situations students are tested on one-minute timings. Expectations of automaticity vary somewhat. Translating a one-second-response time directly into writing answers for one minute would produce 60 answers per minute. However, some children, especially in the primary grades, cannot write that quickly. "In establishing mastery rate levels for individuals, it is important to consider the learner's characteristics (e.g., age, academic skill, motor ability). For most students a rate of 40 to 60 correct digits per minute [25 to 35 problems per minute] with two or few errors is appropriate (Mercer & Miller, 1992, p.23)." This rate of 35 problems per minute seems to be the lowest noted in the literature.

Other authors noted research which indicated that "students who are able to compute basic math facts at a rate of 30 to 40 problems correct per minute (or about 70 to 80 digits correct per minute) continue to accelerate their rates as tasks in the math curriculum become more complex....[however]...students whose correct rates were lower than 30 per minute showed progressively decelerating trends when more complex skills were introduced. The *minimum* correct rate for basic facts should be set at 30 to 40 problems per minute, since this rate has been shown to be an indicator of success with more complex tasks (Miller & Heward, 1992, p. 100)." Rates of 40 problems per minute seem more likely to continue to accelerate than the lower end at 30.

Another recommendation was that "the criterion be set at a rate [in digits per minute] that is about 2/3 of the rate at which the student is able to write digits (Stein et al., 1997, p. 87)." For example a student who could write 100 digits per minute would be expected to write 67 digits per minute, which translates to between 30 and 40 problems per minute. Howell and Nolet (2000) recommend an expectation of 40 correct facts per minute, with a modification for students who write at less than 100 digits per minute. The number of digits per minute is a percentage of 100 and that percentage is multiplied by 40 problems to give the expected number of problems per minute; for example, a child who can only write 75 digits per minute would have an expectation of 75% of 40 or 30 facts per minute.

If measured individually, a response delay of about 1 second would be automatic. In writing 40 seems to be the minimum, up to about 60 per minute for students who can write that quickly. Teachers themselves range from 40 to 80 problems per minute. Sadly, many school districts have expectations as low as 50 problems in 3 minutes or 100 problems in five minutes. These translate to rates of 16 to 20 problems per minute. At this rate answers can be counted on fingers. So this "passes" children who have only developed procedural knowledge of how to figure out the facts, rather than the direct recall of automaticity.

What type of practice effectively leads to automaticity?

Some children, despite a great deal of drill and practice on math facts, fail to develop fluency or automaticity with the facts. Hasselbring and Goin (1988) reported that researchers, who examined the effects computer delivered drill-and-practice programs, have, generally reported that computer-based drill-and-practice seldom leads to automaticity, especially amongst children with learning disabilities.

Ashlock noted that, "the major result of practice is increased rate and accuracy in doing the task that is *actually* practiced. For example, the child who practices counting on his fingers to get missing sums usually learns to count on his fingers more quickly and accurately. The child who uses skip counting or repeated addition to find a product learns to use repeated addition more skillfully—but he continues to add. Practices does not necessarily lead to more mathematically mature ways of finding the missing number or to immediate recall as such....If the process to be reinforced is recalling, then it is important that the child feel secure to state what he recalls, even if he will later check his answer...(1971, p. 363)" Hasselbring et al. stated that, "From our research we have concluded that if a student is using procedural knowledge (i.e., counting strategies) to solve basic math facts, typical computer-based drill and practice activities do not produce a developmental shift whereby the student retrieves the answers from memory (1988, p. 4)." This author experienced the same failure of typical practice procedures to develop fluency in his own students with learning disabilities. Months and sometimes multiple years of practice resulted in students who barely more fluent than they were at the start.

For practice to lead to automaticity, students must be “recalling” the facts, rather than “deriving” them.

Hasselbring et al. noted that the minimal increases in fluency seen in such students

> “can be attributed to students becoming more efficient at counting and not due to the development of automatic recall of facts from memory....We found that if a learner with a mild disability has not established the ability to retrieve an answer from memory before engaging in a drill-and-practice activity then time spend in drill-and-practice is essentially wasted. On the other hand, if a student can retrieve a fact from memory, even slowly, then use of drill-and-practice will quickly lead to the fluent recall of that fact....the key to making computer-based drill an activity that will lead to fluency is through additional instruction that will establish information in long-term memory....Once acquisition occurs (i.e., information is stored in long term memory), then drill-and-practice can be used effectively to make the retrieval of this information fluent and automatic (Hasselbring & Goin, 1988, p. 203).”

Other researchers have found the same phenomena, although they have used different language. If students can recall answers to fact problems rather than derive them from a procedure some educators would say that those answers have been “learned.” Therefore continued practice could be called “overlearning.” For example, “Drill and practice software is most effective in the overlearning phase of learning—i.e., effective drill and practice helps the student develop fast and efficient retrieval processes (Goldman & Pelegrino, 1986).

If students are not recalling the answers and recalling them correctly, accurately, while they are practicing—the practice is not valuable. So teaching students some variety of memory aid would seem to be necessary. If the students can't recall a fact directly or instantly, if they recall the "trick" they can get the right answer—and then continue to practice remembering the right answer.

However, there is another way to achieve the same result. Garnett, while encouraging student use of a variety of relationship strategies actually hinted at the answer when she suggested that teachers should "press for speed (direct retrieval) with a few facts at a time (1992, p. 213)."

How efficient is practice when a small enough set of facts to remember easily is practiced? Logan and Klapp (1991a) found that college students required less than 15 minutes of practice to develop automaticity on a small set of six alphabet arithmetic facts. "The experimental results were predicted by theories that assume that memory is the process underlying automaticity. According to those theories, performance is automatic when it is based on single-step, direct-access retrieval of a solution from memory; in principle, that can occur after a single exposure. (Logan & Klapp, 1991a, p. 180)."

This suggests that if facts were learned in small sets, that could easily be remembered after a couple of presentations, the process of developing automaticity with math facts could proceed with relatively little pain and considerably less drill than is usually associated with learning "all" the facts. "The conclusion that automatization depends on the number of presentations of individual items rather than the total amount of practice has interesting implications. It suggests that automaticity can be attained very quickly if there is not much to be learned. Even if there is much to be learned, parts of it can be automatized quickly if they are trained in isolation (Logan and Klapp, 1991a, p. 193)."

Logan & Klapp's studies show that, "The crucial variable [in developing automaticity] is the number of trials per item, which reflects the opportunity to have memorized the items (1991a, p. 187)." So if children are only asked to memorize a couple of facts at a time, they could develop automaticity fairly quickly with those facts. "Apparently extended practice is not necessary to produce automaticity. Twenty minutes of rote memorization produced the same result as 12 sessions of practice on the task! (Logan, 1991a, p. 355)."

Has this in fact been demonstrated with children? As has been noted earlier few researchers focus on the third stage development of automaticity. And there are even fewer studies that instruct on anything less than 7 to 10 facts at a time. However, in cases where small sets of facts have been used, the results have been uniformly successful, even with students who previously had been "unable" to learn math facts. Cooke and colleagues also found evidence in practicing math facts to automaticity, "suggesting that greater fluency can be achieved when the instructional load is limited to only a few new facts interspersed with a review of other fluent facts (1993, p. 222)." Stein et al. indicate that a "set" of "new facts" should consist of no more than four facts (1997, p. 87). Hasselbring et al. found that,

> "Our research suggest that it is best to work on developing declarative knowledge by focusing on a very small set of new target facts

> at any one time—no more than two facts and their reversals. Instruction on this target set continues until the student can retrieve the answers to the facts consistently and without using counting strategies. …
>
> We begin to move children away from the use of counting strategies by using ‘controlled response times.’ A controlled response time is the amount of time allowed to retrieve and provide the answer to a fact. We normally begin with a controlled response time of 3 seconds or less and work down to a controlled response time around 1.25 seconds.
>
> We believe that the use of controlled response times may be the most critical step to developing automatization. It forces the student to abandon the use of counting strategies and to retrieve answers rapidly from the declarative knowledge network.
>
> If the controlled response time elapses before the child can respond, *the student is given the answer* and presented with the fact again. This continues until the child gives the correct answer within the controlled response time. (1988, p.4).” [emphasis added].

Giving the answer to the student, rather than allowing them to derive the answer changes the nature of the task. Instead of simply *finding the answer* the student is involved in checking to see if he or she *remembers* the answer. If not, the student is reminded of the answer and then gets more opportunities to practice “remembering” the fact’s answer. Interestingly this finding—that it is important to allow the student only a short amount of time before providing the answer—has been studied extensively as a procedure psychologists call “constant time delay.”

Several studies have found that a teaching using a constant time delay procedure for teaching math facts is quite effective (Bezuk & Cegelka, 1995). This “near-errorless technique” means that if the student does not respond within the time allowed, a “controlling prompt” (typically a teacher modeling the correct response) is provided. The student then repeats the task and the correct answer (Koscinski & Gast, 1993b).” The results have shown that this type of near-errorless practice has been effective whether delivered by computer (Koscinski & Gast, 1993a) or by teachers using flashcards (Koscinski & Gast, 1993b).

Two of the aspects of the constant time delay procedures are instructionally critical. One is the time allowed is on the order of 3 or 4 seconds, therefore ensuring that students are using memory retrieval rather than reconstructive procedures, in other words, they are remembering rather than figuring out the answers. Second, if the student fails to “remember” he or she is immediately told the answer and asked to repeat it. So it becomes clear that the point of the task is to “remember” the answers rather than continue to derive them over and over.

These aspects of constant time delay teaching procedures are the same ones that were found to be effective by Hasselbring and colleagues. Practicing on a small set of facts that are recalled from memory until those few facts are answered very quickly contrasts sharply with the typical practice of timed tests on all 100 facts at a time. In addition, the requirement that these small sets are practiced until answers come easily in a matter of one or two seconds, is also unusual. Yet, research indicates that developing very strong associations between each small set of facts and their answers (as shown by quick, correct answers), before moving on to learn more facts will improve success in learning.

Campbell's research found that most difficulty in learning math facts resulted from interference from correct answers to "allied" problems—where one of the factors was the same (Campbell, 1987a; Campbell, 1987b; Campbell & Graham, 1985). He asked the question, "How might interference in arithmetic fact learning be minimized?…If strong correct associations are established for problems and answers encountered early in the learning sequence (i.e. a high criterion for speed and accuracy), those problems should be less susceptible to retroactive interference when other problems and answers are introduced later. Furthermore, the effects of proactive interference on later problems should also be reduced (Campbell, 1987b, p. 119-120)."

Because confusion among possible answers is the key problem in learning math facts, the solution is to establish a mastery-learning paradigm, where small sets of facts are learned to high levels of mastery, before adding any more facts to be learned. This is what researchers have found, when they have focused on the development of automaticity. The mechanics of achieving this gradual mastery-learning paradigm have been outlined by two sets of authors. Hasselbring et al. found that,

"As stated, the key to making drill and practice an activity that will lead to automaticity in learning handicapped children is additional instruction for establishing a declarative knowledge network. Several instructional principles may be applied in establishing this network:

1. Determine learner's level of automaticity.

2. Build on existing declarative knowledge.
3. Instruct on a small set of target facts.
4. Use controlled response times.
5. Intersperse automatized with targeted nonautomatized facts during instruction.

(1988, p. 4)."

The teacher cannot focus on a small set of facts and intersperse them with facts that are already automatic until after an assessment determines which facts are already automatic. Facts that are answered without hesitation after the student reads them aloud would be considered automatic. Facts that are read to the student should be answered within a second.

Silbert, Carnine, and Stein recommended similar make-up for a set of facts for flashcard instruction. "Before beginning instruction, the teacher makes a pile of flash cards. This pile includes 15 cards: 12 should have the facts the student knew instantly on the tests, and 3 would be facts the student did not respond to correctly on the test. (1990, p. 127)."

These authors also recommended procedures for flashcard practice similar to the constant time delay teaching methods. "If the student responds correctly but takes longer than 2 seconds or so, the teacher places the card back two or three cards from the front of the pile. Likewise, if the student responds incorrectly, the teacher tells the student the correct answer and then places the card two or three cards back in the pile. Cards placed two or three back from the front will receive intensive review. The teacher would continue placing the card two or three places back in the pile until the student responds acceptably (within 2 seconds) four times in a row (Silbert et al., 1990, p. 130)."

Hasselbring et al. described how their computer program “Fast Facts” provided a similar, though slightly more sophisticated presentation order during practice. “Finally, our research suggests that it is best to work on developing a declarative knowledge network by interspersing the target facts with other already automatized facts in a prespecified, expanding order. Each time the target fact is presented, another automatized fact is added as a ‘spacer’ so that the amount of time between presentations of the target fact is expanded. This expanding presentation model requires the student to retrieve the correct answers over longer and longer periods (1988, p. 5).”

Unfortunately the research-based math facts drill-and-practice program “Fast Facts” developed by Hasselbring and colleagues was not developed into a commercial product. Nor were their recommendations incorporated into the design of existing commercial math facts practice programs, which seldom provide either controlled response times or practice on small sets. What practical alternatives to cumbersome flashcards or ineffective computer drill-and-practice programs are available to classroom teachers who want to use the research recommendations to develop automaticity with math facts?

How can teachers implement class-wide effective facts practice?

Stein et al. tackled the issue of how teachers could implement an effective facts practice program consistent with the research. As an alternative to total individualization, they suggest developing a sequence of learning facts through which all students would progress. Stein et al. describe the worksheets that students would master one at a time.

> "Each worksheet would be divided into two parts. The top half of the worksheets should provide practice on new facts including facts from the currently introduced set and from the two preceding sets. More specifically, each of the facts from the new set would appear four times. Each of the facts from the set introduced just earlier would appear three times, and each of the facts from the set that preceded that one would appear twice….The bottom half of the worksheet should include 30 problems. Each of the facts from the currently introduced set would appear twice. The remaining facts would be taken from previously introduced sets. (1997, p. 88)."

The daily routine consists of student pairs, one of which does the practicing while the other follows along with an answer key. Stein et al. specify:

"The teacher has each student practice the top half of the worksheet twice. Each practice session is timed…the student practices by saying complete statements (e.g., 4 + 2 = 6), rather than just answers….If the student makes an error, the tutor corrects by saying the correct statement and having the student repeat the statement. The teacher allows students a minute and a half when practicing the top part and a minute when practicing the bottom half. (1997, p. 90)."

After each student practices the teacher conducts a timed one-minute test of the 30 problems on the bottom half. Students who answered correctly at least 28 of the 30 items within the minute allowed on the bottom half test have passed the worksheet and are then given the next worksheet to work on. A specific performance criterion for fact mastery is critical to ensure mastery before moving on to additional material. These criteria correspond to the slowest definition of automaticity in the research literature. Teachers and children could benefit from higher rate of mastery before progressing, in the range of 40 to 60 problems per minute—if the students can write that quickly.

Stein et al. also delineate the organizational requirements for such a program to be effective and efficient: "A program to facilitate basic fact memorization should have the following components:

1. a specific performance criterion for introducing new facts.
2. intensive practice on newly introduced facts
3. systematic practice on previously introduced facts
4. adequate allotted time
5. a record-keeping system
6. a motivation system (1997, p. 87)."

The worksheets and the practice procedures ensure the first three points. Adequate allotted time would be on the order of 10 to 15 minutes per day for each student of the practicing pair to get their three minutes of practice, the one-minute test for everyone and transition times. The authors caution that "memorizing basic facts may require months and months of practice (Stein et al., 1997, p. 92)." A record-keeping system could simply record the number of tries at each worksheet and which worksheets had been passed. A motivation system that gives certificates and various forms of recognition along the way will help students maintain the effort needed to master all the facts in an operation.

Such a program of gradual mastery of small sets of facts at a time is fundamentally different than the typical kind of facts practice. Because children are learning only a small set of new facts it does not take many repetitions to commit them to memory. The learning is occurring during the "practice" time. Similarly because the timed tests are only over the facts already brought to mastery, children are quite successful. Because they see success in small increments after only a couple of days practice, students remain motivated and encouraged.

Contrast this with the common situation where children are given timed-tests over all 100 facts in an operation, many of which are not in long term memory (still counting). Students are attempting to become automatic on the whole set at the same time. Because children's efforts are not focused on a small set to memorize, students often just become increasingly anxious and frustrated by their lack of progress. Such tests do not teach students anything other than to remind them that they are unsuccessful at math facts. Even systematically taking timed tests daily on the set of 100 facts is ineffective as a teaching tool and is quite punishing to the learner. "Earlier special education researchers attempted to increase automaticity with math facts by systematic drill and practice...But this "brute force" approach made mathematics unpleasant, perhaps even punitive, for many (Gersten & Chard, 1999, p. 21)."

These inappropriate type timed tests led to an editorial by Marilyn Burns in which she offered her opinion that, "Teachers who use timed tests believe that the tests help children learn basic facts....Timed tests do not help children learn (Burns, 1995, p. 408-409)." Clearly timed tests only establish whether or not children have learned—they do not teach. However, if children are learning facts in small sets, and are being taught their facts gradually, then timed tests will demonstrate this progress. Under such circumstances, if children are privy to graphs showing their progress, they find it quite motivating (Miller, 1983).

Given the most common form of timed tests, it is not surprising that Isaacs and Carroll argue that "...an over reliance on timed tests is more harmful than beneficial (Burns, 1995), this fact has sometimes been misinterpreted as meaning that they should never be used. On the contrary, if we wish to assess fact proficiency, time is important. Timed tests also serve the important purpose of communicating to students and parents that basic-fact proficiency is an explicit goal of the mathematics program. However, daily, or even weekly or monthly, timed tests are unnecessary (1999, p. 512)."

In contrast, when children are successfully learning the facts, through the use of a properly designed program, they are happy to take tests daily, to see if they've improved. "Results from classroom studies in which time trials have been evaluated show just the opposite: Accuracy does not suffer, but usually improves, and students enjoy being timed...When asked which method they preferred, 26 of the 34 students indicated they liked time trials better than the untimed work period (Miller & Heward, 1992, p. 101-102)."

In summary, students can develop their skill with math facts past strategies for remembering facts into automaticity, or direct retrieval of math fact answers. Automaticity is achieved when students can answer math facts with no hesitation or no more than one second delay—which translates into 40 to 60 problems per minute. What is required for students to develop automaticity is a particular kind of practice focused on small sets of facts, practiced under limited response times, where the focus is on remembering the answer quickly rather than figuring it out. The introduction of additional new facts should be withheld until students can demonstrate automaticity with all previously introduced facts. Under these circumstances students are successful and enjoy graphing their progress on regular timed tests.

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