Third stage of Learning Math Facts: Developing automaticity

The third stage of learning math facts has been called mastery, or overlearning, or the development of automaticity. In this stage children develop the capacity to simply recall the answers to facts without resorting to anything other direct retrieval of the answer. Ashlock (1971) indicated that children must have "immediate recall" of the basic facts so they can use them "with facility" in computation. "If not, he may find it difficult to develop skill in computation for he will frequently be diverted into tangential procedures (1971, p. 363)."

When automaticity is developed, one of its most notable traits is speed of processing. "Proficient levels of performance go beyond the accuracy (quality) of an acquired skill to encompass sufficient speed (quantity) of performance. It is this sort of proficiency with basic facts, rather than accuracy per se, which is so notably lacking in many learning disabled children's computation performance (Garnett & Fleischner, 1983, p. 224)."

The development of automaticity has been studied extensively in the psychology literature. Research by psychologists on math facts and other variations on automatic fact retrieval has been extensive, probably because "Children's acquisition of skill at mental addition is a paradigm case of automaticity (Logan & Klapp, 1991a, p.180).

While research continues to refine the details of the models of the underlying processes, there has for some time been a clear consensus that adults "retrieve" the answers to math facts directly from memory. The "modern theories that argue that the process underlying automaticity is memory retrieval: According to these theories, performance is automatic when it is based on direct-access, single-step retrieval of solutions from memory rather than some algorithmic computation (Logan & Klapp, 1991a, p. 179)." When facts have been well practiced, they are "remembered" quickly and automatically—which frees up other mental processes to use the facts in more complex problems (Ashcraft, 1992; Campbell, 1987b; Logan, 1991a).

Baroody made one of the last forceful defenses of the alternative model that adults continue to use, albeit very quickly, "rules, procedures, or principles from which a whole range of combinations could be reconstructed (1985, p. 95)." Baroody's examples of how we remember the + 0 facts or the + 1 facts are now understood to be exceptions to the general rule of simple retrieval from memory. Baroody then took the exceptions to make a rule and went on to say that, "According to this alternative model, 'mastery of the facts' would *include* discovering, labeling, and internalizing *relationships*. Meaningful instruction (the teaching of thinking strategies) would probably contribute more directly to this process than drill approach alone (1985, p. 95)." This did not include any empirical examinations of his theory.

Baroody's notion suggests that additional work to memorize math facts is not necessary as long as students can reconstruct facts from their relationships with other known facts. However, a variety of careful psychological experiments have demonstrated that adults do not use procedures or algorithms to derive the answers to math facts—they simply remember or retrieve them (Ashcraft, 1985; Campbell, 1987a; Campbell, 1987b; Campbell & Graham, 1985; Graham, 1987; Logan, 1988; Logan & Klapp, 1991a; Logan & Klapp, 1991b; McCloskey, Harley, & Sokol, 1991).

For example, Logan and Klapp (1991a) did experiments with "alphabet arithmetic" where each letter corresponded to a number (A=1, B=2, C=3, etc.). In these studies college students initially experienced the same need to count to figure out sums that elementary children did. In the initial-counting stage the time to answer took longer the higher the addends (requiring more time to count up). However, once these "alphabet facts" became automatic—they took the same amount of time regardless of the addends, and they took less time than they could possibly have counted. In addition, they found that when students were presented with new items they had not previously had—they reverted to counting and the time to answer went up dramatically and was again related to the size of the addends. When the students were presented with a mix of items, some which required counting and others, which they had already learned, they were still able to answer the ones they had previously learned roughly as fast as they had before. Response times were able to clearly distinguish between the processes used on facts the students were "figuring out" vs. the immediate retrieval process for facts that had become automatic.

Does the same change in processes apply to children as they learn facts? Another large set of experiments has demonstrated that as children mature they transition from procedural (counting) methods of solving problems to the pattern shown by adults of direct retrieval. (Ashcraft, 1982; Ashcraft, 1984; Ashcraft, 1992; Campbell, 1987; Campbell & Graham, 1985; Geary & Brown, 1991; Graham, 1987; Koshmider & Ashcraft, 1991; Logan, 1991b; Pelegrino & Goldman, 1987). Despite the preponderance of clear evidence that adults and children (after they have learned math facts to automaticity) use direct retrieval—"just remembering"—some writers in respected education journals continue to advance Baroody's theory that "although many facts become automatic, adults also use strategies and rules for certain facts (Isaacs & Carroll, 1999, p. 514)." This phenomena is limited to facts adding + 0 or multiplying x 0 or x 1 (Ashcraft, 1985) and therefore does not indicate adult use of such strategies and rules for any of the rest of the facts. Unfortunately it appears as if some of those who recommend teaching strategies and relationships in the second stage of math facts see it primarily as a way to de-emphasize the memorization needed for automatization of math facts (Isaacs & Carroll, 1999).

Interestingly, most of the research celebrating "strategies and rules" relates to addition and subtraction facts. Once children are expected to become fluent in recalling multiplication facts such strategies are inadequate to the task, and memorization seems to be the only way to accomplish the goal of mastery. Campbell and Graham's (1985) study of error patterns in multiplication facts by children demonstrated that as children matured errors became limited to products related correctly to one or the other of the factors, such as 4 X 7 = 24. See also (Campbell, 1987b; Graham, 1987). They concluded that "the acquisition of simple multiplication skill is well described as a process of associative bonding between problems and candidate answers....determined by the relative strengths of correct and competing associations ... not a consequence ... of the execution of reconstructive arithmetic procedures (Campbell & Graham, 1985, p. 359)." In other words, we remember answers based on practice with the facts rather than using some procedure to derive the answers. And we have difficulty learning them because we recall answers to other similar math fact problems and sometimes fail to correctly discard these incorrect answers (Campbell, 1987b).

Graham (1987) reported on a study where students who learned multiplication facts in a mixed order had little or no difference between response times to smaller and larger fact problems. Graham's recommendation was that facts be arranged in sets where commonly confused problems are not taught together. In addition he suggested, "it may be beneficial to give the more difficult problems (e.g. 3 x 8, 4 x 7, 6 x 9, and 6 x 7) a head start. This could be done by placing them in the first set and requiring a strict performance criterion before moving on to new sets (Graham, 1987, p. 139)."

Ashcraft has studied the development of mental arithmetic extensively (Ashcraft, 1982; Ashcraft, 1984; Ashcraft, 1992; Koshmider & Ashcraft, 1991). These studies have examined response times and error rates over a range of ages. The picture is clear. In young children who are still counting, response rates are consistent with counting rates. Ashcraft found that young children took longer to answer facts—long enough to count the answers. He also found that they took longer in proportion to the sum of the addends—an indication that they were counting. After children learn the facts, the picture changes. Once direct retrieval is the dominant mode of answering, response rates decrease from over 3 seconds down to less than one second on average. Ashcraft notes that other researchers have demonstrated that while young children use a variety of approaches to come up with the answer (as they develop knowledge of the math facts), eventually these are all replaced by direct retrieval in 5th and 6th grade normally achieving children.

Ashcraft and Christy (1995) summarized the model of math fact performance from the perspective of cognitive psychology. "...the whole-number arithmetic facts are eventually stored in long-term memory in a network-like structure .... at a particular level of strength, with strength values varying as a function of an individual's practice and experience....because these variables determine strength in memory (Ashcraft & Christy, 1995, p. 398)." Interestingly enough, Ashcraft and Christy's (1995) examination of math texts led to the discovery in both addition and multiplication that smaller facts (2s-5s) occur about twice as frequently as larger facts (6s-9s). This may in part explain why larger facts are less well learned and have slower response times.

Logan's (1991b) studies of the transition to direct retrieval show that learners start the process of figuring out the answer (procedural knowledge) and use retrieval if the answer is remembered (declarative knowledge) before the process is completed. For a time there is a mixture of some answers being retrieved from memory while others are being figured out. Once items are learned to the automatic level, the answer "occurs" to the learner before they have time to count. Geary and Brown (1991) found the same sort of evolving mix of strategies among gifted, normal, and math-disabled children in the third and fourth grades.

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